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A Law of Large Numbers for the absorbed mass of super Brownian motion with immigration

Yaping Zhu
School of Mathematical Sciences, Beijing Normal University, China
zhuyp@mail.bnu.edu.cn

Abstract
We prove a Law of Large Numbers for the absorbed mass of critical super Brownian motion with immigration, which leads to a constant determined by the immigration measure and the position of barrier.

Introduction

Super Brownian motion have been extensively studied concerning various problems, for example, Li(1995) discussed the immigration diffusion process of the super Brownian motion with absorption over $(0,\infty)$ and derived a stochastic partial differential equation. Li(1999), Hong and Li(1999) proved the central limit theorem for the critical super Brownian motion, and the super Brownian motion with immigration governed by another super Brownian, The limit theorem leads to Gaussian random fields. Recently, Kyprianou, Murillo and Pérez(2013) studied the distribution of the absorbed mass at criticality by the backbone decomposition of supercritical super Brownian motion with a barrier, they mainly used the theory in Dynkin(1991), (1993) and the conclusion in Maillard(2011) about the number of absorbed individuals in supercritical branching Brownianmotion with a barrier.

In this paper, we aim at the absorbed mass of critical super Brownian motion with immigration and prove a Law of Large Numbers for the absorbed mass.

Model Description

Suppose that $X=\{X_t : t\geq 0\}$ is a super-diffusion with motion corresponding to a Brownian motion $\xi=\{\xi_t : t\geq 0\}$, and branching mechanism ψ taking the form

$$\psi(s,x,\lambda)=-a(s,x)\lambda+b(s,x)\lambda^2+\int_{(0,\infty)}(e^{-\lambda u}-1+\lambda u)\eta(s,x,\mathrm{d}u),\quad \lambda\geq 0, \tag{1}$$

where a is a bounded function on $[0,\infty)\times\mathbf{R}$, b is a bounded positive function on $[0,\infty)\times\mathbf{R}$ and η is a measure concentrated on $(0,\infty)$ which satisfies $\int_{(0,\infty)}(u\wedge u^2)\eta(s,x,\mathrm{d}u)<\infty$.

For each $t\geq 0$, $y>0$, define the time-space domain $D_y^t=[0,t)\times(-\infty,y)$. According to Dynkin's theory of exit measures, it is possible to describe the mass in the superprocess X as it first exits the domain D_y^t.

The random measure $X_{D_y^t}$ is supported on $\partial D_y^t=\big(\{t\}\times[-\infty,y)\big)\cup\big([0,t)\times\{y\}\big)$ and is characterized by the Laplace functional

$$\mathbb{P}_\mu\big(e^{-\langle f,X_{D_y^t}\rangle}\big)=e^{-\langle v^t,\mu\rangle}, \tag{2}$$

where $x<y$, $f\in\mathbf{C}_b([0,\infty)\times\mathbf{R})$, $\mu\in\mathcal{M}_F((0,\infty)\times\mathbf{R})$ and v^t is the unique non-negative solution for the equation

$$v^t(r,x)+\Pi_{r,x}\int_r^{\tau_{D_y^t}}\psi(v^t)(s,\xi_s)ds=\Pi_{r,x}f(\tau_{D_y^t},\xi_{\tau_{D_y^t}}), \tag{3}$$

where $\tau_{D_y^t}=\inf\{t>0:(t,\xi_t)\notin[0,t)\times(-\infty,y)\}$, for every (r,x), $\Pi_{r,x}(\xi_r=x)=1$. Let $\tau_{D_y}=\inf\{t>0:\xi_t\notin(-\infty,y)\}$, it is easy to know $\tau_{D_y^t}=\tau_{D_y}\wedge t$.

In the next sections, we consider the following branching mechanism $\psi(s,x,\lambda)=b(s,x)\lambda^2$ and $\mu(ds,dx)=\mathbf{1}_{\{s=0\}}\mathrm{d}sm(\mathrm{d}x)+\mathbf{1}_{\{s>0\}}\mathrm{d}s\gamma(\mathrm{d}x)$, where m,γ are both finite measure on $\mathbf{R}$ with compact support.

Main results

A Law of Large Numbers for One-dimension

Theorem 1. *For each $f\in\mathbf{C}_b^+((-\infty,y])$, $\langle f,\frac{1}{t}X_{D_y^t}\rangle$ convergence in probability to $\langle\Pi_{0,\cdot}f(\xi_{\tau_{D_y}}),\gamma\rangle$ when $t\to\infty$.*

Remark 1. *Actually, $\langle\Pi_{0,\cdot}f(\xi_{\tau_{D_y}}),\gamma\rangle=f(y)\langle 1,\gamma\rangle$ is a positive finite constant.*

The main idea of the Proof: In order to prove the large number law, the order of the first moment should be estimated and the limit of the second moment should be analyzed.

Based on the above assumptions about ψ and μ, using the Feynman—Kac formula, we can get

$$\mathbb{P}_\mu\langle f,X_{D_y^t}\rangle=\langle\Pi_{0,\cdot}f(\xi_{\tau_{D_y^t}}),m\rangle+\int_0^t\langle\Pi_{s,\cdot}f(\xi_{\tau_{D_y^t}}),\gamma\rangle\mathrm{d}s,$$

which implies that the order of the absorbed mass is equal to t when $t\to\infty$. Because m,γ are both finite measure with compact support and f is a bounded function ,

$$\lim_{t\to\infty}\mathbb{P}_\mu\langle f,\frac{1}{t}X_{D_y^t}\rangle=\langle\Pi_{0,\cdot}f(\xi_{\tau_{D_y^t}}),\gamma\rangle.$$

In order to calculate the second moment, we need to calculate I_1 and I_2, where

$$I_1=\lim_{t\to\infty}\frac{1}{t^2}\langle\Pi_{0,\cdot}\int_0^{\tau_{D_y^t}}b(s,\xi_s)\mathrm{d}s,m\rangle,$$

$$I_2=\lim_{t\to\infty}\frac{1}{t^2}\int_0^t\langle\Pi_{s,\cdot}\int_s^{\tau_{D_y^t}}b(r,\xi_r)\mathrm{d}r,\gamma\rangle\mathrm{d}s,$$

According to the definition of $\tau_{D_y^t}$, it is easy to know $I_1=0$.

When prove $I_2=0$, we use the fact

$$\Pi_{0,x}(\tau_{D_y}<t)=\int_0^t\frac{y-x}{\sqrt{2\pi u^3}}e^{-\frac{(y-x)^2}{2u}}du.$$

A Law of Large Numbers for High-dimension

We can also consider high-dimension situations, suppose that $Y=\{Y_t : t\geq 0\}$ is a super diffusion with motion corresponding to a d dimension Brownian motion $B(t)=\{B_1(t),B_2(t),\cdots,B_d(t)\}$, where $d\geq 2$ and branching mechanism ψ also taking the form

$$\psi(s,x,\lambda)=b(s,x)\lambda^2,\quad\lambda\geq 0,$$

where b is a bounded positive function on $[0,\infty)\times\mathbf{R}^d$.

Similarly,we can define the time-space domain $G_y^t=[0,t)\times(-\infty,y)\times\mathbf{R}^{d-1}$ and the exit measures process $Y_{G_y^t}$ which is characterized by the Laplace functional

$$\mathbb{P}_\mu\big(e^{-\langle f,Y_{G_y^t}\rangle}\big)=e^{-\langle u^t,\mu\rangle}, \tag{4}$$

where$f\in\mathbf{C}_b([0,\infty)\times\mathbf{R}^d)$, $\mu\in\mathcal{M}_F((0,\infty)\times\mathbf{R}^d)$ is defined as above and u^t is the unique non-negative solution for the equation

$$u^t(r,x)+\Pi_{r,x}\int_r^{\tau_{G_y^t}}b(s,B(s))v^t\big(s,B(s)\big)^2\mathrm{d}s=\Pi_{r,x}f\big(\tau_{G_y^t},B(\tau_{G_y^t})\big), \tag{5}$$

where $x=(x_1,x_2,...,x_d)\in\mathbf{R}^d$, $x_1<y$, $\tau_{G_y^t}=\inf\{t>0:(t,B_1(t))\notin[0,t)\times(-\infty,y)\}$. Let $\tau_{G_y}=\inf\{t>0:B_1(t))\notin(-\infty,y)\}$. We can understand the above process with the help of the following figure

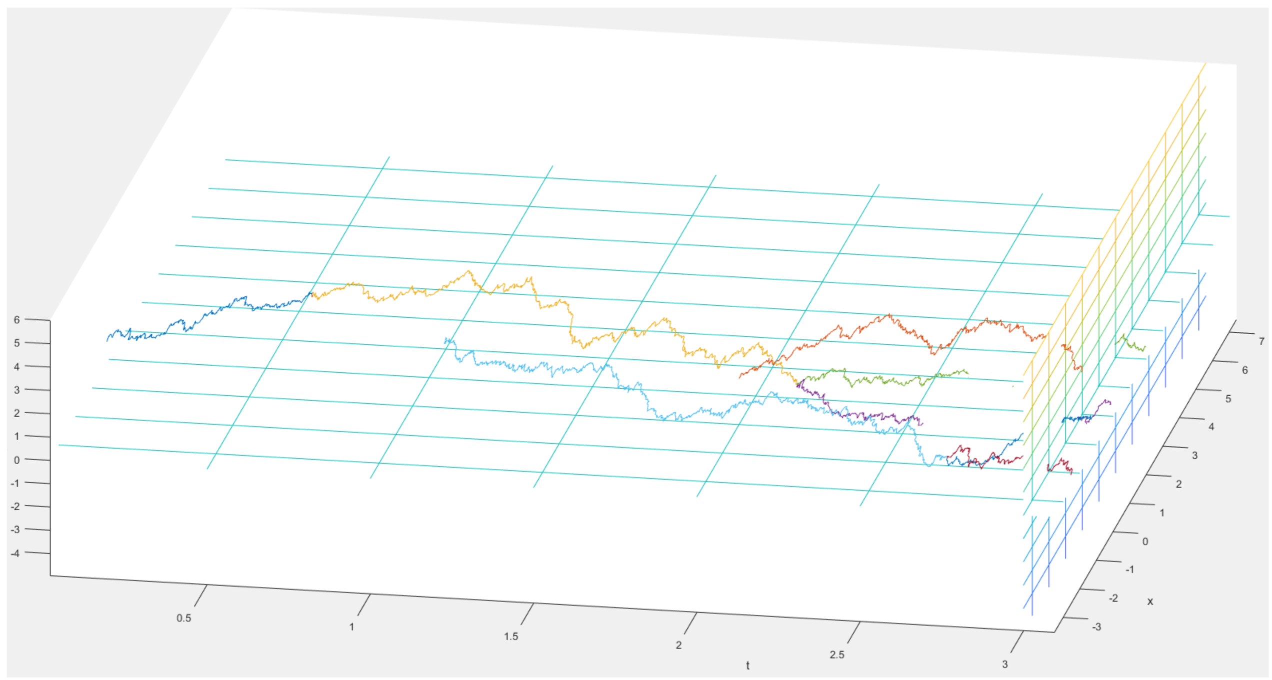

Figure 1: High-dimension

Theorem 2. *For each $f\in\mathbf{C}_b^+((-\infty,y]\times\mathbf{R}^{d-1})$, $\langle f,\frac{1}{t}Y_{G_y^t}\rangle$ convergence in probability to $\langle\Pi_{0,\cdot}f(B(\tau_{G_y})),\gamma\rangle$ when $t\to\infty$.*

Remark 2. *Actually, $\langle\Pi_{0,\cdot}f(B(\tau_{G_y})),\gamma\rangle$ is a positive finite constant determined by the following formula*

$$\langle\Pi_{0,\cdot}f(B(\tau_{G_y})),\gamma\rangle=\int_{\mathbf{R}^d}\int_{\mathbf{R}^{d-1}}f(y,z_2,...,z_d)\prod_{i=2}^d h_i(z_i)dz_2\cdots dz_d\gamma(dx_1,...,dx_d), \tag{6}$$

where $h_i(z)=\frac{y-x_1}{\pi[(y-x_1)^2+(z-x_i)^2]}$, $2\leq i\leq d$.

Further Research

We can also consider the absorbed mass with more general drift for supercritical super Brownian motion and the total mass conditioned on the extinction time is equal to infinite. Further, we can consider the limit behavior of the cumulative semigroup of super Brownian motion with a barrier.

References

[1] DYNKIN, E.B. (1991). A probabilistic approach to one class of nonlinear differential equations. *Probab. Theory Related Fields*. 89, 89-115.

[2] DYNKIN, E.B. (1993). Superprocesses and partial differential equations. *Ann. Probab*. 21, 1185-1262.

[3] Hong, W.M., Li, Z.H. (1999). A central limit theorem for super Brownian motion with super Brownian immigration. *Journal of Applied Probability*, 36(4), 1218-1224.

[4] Kyprianou, A.E., Murillo-Salas, A., & J.L Pérez. (2011). An application of the backbone decomposition to supercritical super-Brownian motion with a barrier. *Journal of Applied Probability*, 49(3).

[5] Li, Z.H., Shiga T. (1995). Measure-valued branching diffusions: immigrations, excursions and limit theorems. *Journal of Mathematics of Kyoto University* 35(2), 233-274.

[6] Li, Z.H. (1999). Some central limit theorems for super Brownian motions. *Acta Mathematica Scientia (English Edition)*, 19(2), 121-126.

[7] Maillard, P. (2013). The number of absorbed individuals in branching Brownian motion with a barrier. *Annales De Linstitut Henri Poincare Probability & Stats*, 49(2), 428-455.

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